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Majorana Transformation for Differential Equations

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We present a method for reducing the order of ordinary differential equations satisfying a given scaling relation (Majorana scale-invariant equations). We also develop a variant of this method, aimed to reduce the degree of nonlinearity of the lower order equation. Some applications of these methods are carried out and, in particular, we show that second-order Emden–Fowler equations can be transformed into first-order Abel equations. The work presented here is a generalization of a method used by Majorana in order to solve the Thomas–Fermi equation.

KEY WORDS: Majorana scale-invariant differential equations; Emden–Fowler equations; solution of the Thomas–Fermi equation.

1. INTRODUCTION

In a recent paper (Esposito, in press) we have described a method, originally due to Majorana (Esposito *et al.*, in press), able to give the series solution of the Thomas–Fermi equation (with appropriate boundary conditions) through only one quadrature. Such a method, giving a (semianalytic) parametric solution of the considered equation, is based on a particular double change of variables which transforms the second-order Thomas–Fermi equation into a first-order equation, whose solution is then obtained by series expansion.

Here we show that the transformation method, used by Majorana in that particular case, applies to a large class of ordinary differential equations as well, and prove a simple but general theorem for reducing the order of these equations.

The Majorana idea is a straigthforward generalization of known concepts and to show this we briefly recall, in the following section, some definitions and peculiarities of particular differential equations. In Section 3 we then introduce a new class of differential equations and give the method for reducing the order of

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such equations. In Section 4 a variant of this method is presented and in Section 5 some applications are reported which are particularly relevant in mathematical physics.

2. PRELIMINARIES

Let us consider a general differential equation of order n in the independent variable x and dependent one y:

$$F\left(x, y, y', y'', \ldots, y^{(n)}\right) = 0, \tag{1}$$

where a prime denotes differentiation with respect to x.

Equation (1) is said to be an *autonomous equation* if the variable x does not appear explicitly:

$$F\left(x, y, y', y'', \ldots, y^{(n)}\right) = F\left(y, y', y'', \ldots, y^{(n)}\right). \tag{2}$$

In such a case, by changing the set of variables from $(x, y(x))$ to a novel one $(y, u(y))$ through

$$\begin{aligned} y' &= u(y) \\ y'' &= u(y)\frac{du(y)}{dy}, \end{aligned} \tag{3}$$

where $u(y)$ is a given function of y, the considered differential equation can always be reduced to an equation of order $n-1$ in the independent variable y and dependent one u (Bender and Orszag, 1978; Polyamin and Zaitsev, 1995):

$$G\left(y, u, \frac{du}{dy}, \frac{d^2u}{dy^2}, \ldots, \frac{d^{n-1}u}{dy^{n-1}}\right) = 0. \tag{4}$$

The differential equation (1) is, instead, *equidimensional-in-x* if it is invariant under the transformation $x \to \alpha x$ for any $\alpha \neq 0$:

$$F\left(\alpha x, y, \alpha^{-1}y', \alpha^{-2}y'', \ldots, \alpha^{-n}y^{(n)}\right) = F\left(x, y, y', y'', \ldots, y^{(n)}\right). \tag{5}$$

This equation can be transformed (Bender and Orszag, 1978; Polyamin and Zaitsev, 1995) into an autonomous equation in the variables $(z, y(z))$:

$$G\left(y, \frac{dy}{dz}, \frac{d^2y}{dz^2}, \ldots, \frac{d^ny}{dz^n}\right) = 0, \tag{6}$$

by changing the independent variable:

$$\begin{aligned} x &= e^z \\ x\frac{d}{dx} &= \frac{d}{dz}. \end{aligned} \tag{7}$$

$$\vdots$$

Thus, equidimensional-in-x equations of order n can always be reduced to differential equations of order $n - 1$.

Scale-invariant equations satisfy the property:

$$F\left(\alpha x, \alpha^c y, \alpha^{c-1} y', \alpha^{c-2} y'', \ldots, \alpha^{c-n} y^{(n)}\right) = F\left(x, y, y', y'', \ldots, y^{(n)}\right) \qquad (8)$$

(i.e. they are invariant for $x \to \alpha x$, $y \to \alpha^c y$) for any value of $\alpha \neq 0$ and some value c, and they can be transformed into equidimensional-in-x equations (Bender and Orszag, 1978; Polyamin and Zaitsev, 1995):

$$G\left(x, u(x), \frac{du}{dx}, \frac{d^2u}{dx^2}, \ldots, \frac{d^n u}{dx^n}\right) = 0, \qquad (9)$$

$$\vdots$$

with G obeying Eq. (5), by performing the following change of the dependent variable:

$$y(x) = x^c u(x). \qquad (10)$$

Even in this case, scale-invariant equations of order n can be thus reduced to equations of order $n - 1$.

Finally, differential equations which are invariant under the transformation $y \to \alpha y$ for any $\alpha \neq 0$ are said to be *equidimensional-in-y* or *homogeneous* equations:

$$F\left(x, \alpha y, \alpha y', \alpha y'', \ldots, \alpha y^{(n)}\right) = F\left(x, y, y', y'', \ldots, y^{(n)}\right). \qquad (11)$$

By changing the dependent variable through

$$y(x) = e^{u(x)}, \qquad (12)$$

it can be transformed into an equation of order $n - 1$ in the variables $(x, u(x))$ (Bender and Orszag, 1978; Polyamin and Zaitsev, 1995):

$$G\left(x, u(x), \frac{du}{dx}, \frac{d^2u}{dx^2}, \ldots, \frac{d^{n-1}u}{dx^{n-1}}\right) = 0. \qquad (13)$$

Thus, we know that the order of a given differential equation can always be reduced by one unit if this belongs to one of the four different classes mentioned above.

3. MAJORANA TRANSFORMATION

For future convenience we now consider scale-invariant equations from a different point of view and introduce another class of differential equations. Equation (1) is said to be *Majorana scale-invariant* if it is invariant for $x \to \alpha^c x$, $y \to \alpha y$ for any $\alpha \neq 0$ and some value c:

$$F\left(\alpha^c x, \alpha y, \alpha^{1-c} y', \alpha^{1-2c} y'', \ldots, \alpha^{1-nc} y^{(n)}\right) = F\left(x, y, y', y'', \ldots, y^{(n)}\right). \qquad (14)$$

It is easy to prove the following proposition: *a Majorana scale-invariant equation of order n can always be reduced to a differential equation of order* $n - 1$. In fact, by changing the role of the dependent and the independent variables $x \to y$, $y \to x$, Eq. (1) can be transformed into

$$G\left(y, x(y), \frac{dx}{dy}, \frac{d^2x}{dy^2}, \ldots, \frac{d^nx}{dy^n}\right) = 0, \tag{15}$$

where now G satisfies the condition (8):

$$G\left(\alpha y, \alpha^c x, \alpha^{c-1}\frac{dx}{dy}, \alpha^{c-2}\frac{d^2x}{dy^2}, \ldots, \alpha^{c-n}\frac{d^nx}{dy^n}\right) = G\left(y, x(y), \frac{dx}{dy}, \frac{d^2x}{dy^2}, \ldots, \frac{d^nx}{dy^n}\right), \tag{16}$$

that is Eq. (15) is scale-invariant and the order can be reduced by one unit.

The concept introduced above is not really a new one, since it is related to that of scale-invariant equations. However, the reformulation of the problem in these terms is useful for developing the method for the implementation of order reduction, which is a generalization of that used by Majorana in the framework of the Thomas–Fermi equation (Esposito, in press).

We now describe in detail such a method, which carries out the solution of Eq. (1), with F satisfying Eq. (14), as given in parametric form:

$$\begin{cases} x = x(t) \\ y = y(t). \end{cases} \tag{17}$$

Let us assume that x in Eq. (17) depends on the parameter t through the function $y(t)$ and, eventually, on t itself:

$$x = x(t, y) = x(t, y(t)) = x(t). \tag{18}$$

Since $(x(t), y(t))$ in Eq. (17) is a solution of the considered differential equation (1), supposed to be Majorana scale-invariant, Eqs. (17) and (18) must satisfy the relation (14), meaning that for any $\alpha \neq 0$ and a given value c we have

$$x(t, \alpha y) = \alpha^c x(t, y). \tag{19}$$

This implies that $x(t, y)$ should be an homogeneous function of y:

$$x(t, y) = x(t, 1)y^c \equiv zy^c, \tag{20}$$

where $z = z(t)$ can be considered as an arbitrary but given function of the parameter t. Note that, in such a way, the only unknown function to be determined in order to satisfy Eq. (1) is $y(t)$, and the parametric solution (17) can be rewritten, after Eq. (20), as

$$\begin{cases} x = z(t)y^c(t) \\ y = y(t), \end{cases} \tag{21}$$

with $z(t)$ an arbitrary but given function of t and c is determined by Eq. (14).

We now have to translate the differential equation (1) for $y(x)$ into an equation for $y(t)$ in (21). In the following, differentiation with respect to t will be denoted with a dot, while a prime refers to differentiation with respect to x as above. The t-derivatives of x are, from Eq. (21), as follows:

$$\begin{aligned}
\dot{x} &= \left(\dot{z} + cz\frac{\dot{y}}{y}\right) y^c \equiv x_1(t, y, \dot{y}) \\
\ddot{x} &= \left\{\ddot{z} + 2c\dot{z}\frac{\dot{y}}{y} + cz\left[(c-1)\left(\frac{\dot{y}}{y}\right)^2 + \frac{\ddot{y}}{y}\right]\right\} \equiv x_2(t, y, \dot{y}, \ddot{y}) \\
&\vdots \\
\overset{n\cdot}{x} &= \cdots = x_n(t, y, \dot{y}, \ddot{y}, \ldots, \overset{n\cdot}{y}).
\end{aligned} \tag{22}$$

Using these expressions we can obtain the x-derivatives of y, which are present in Eq. (1), in terms of t, y, and its t-derivatives:

$$\begin{aligned}
y' &= \frac{\dot{y}}{\dot{x}} \equiv y_1(t, y, \dot{y}) \\
y'' &= \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\dot{x}^3} \equiv y_2(t, y, \dot{y}, \ddot{y}) \\
&\vdots \\
y^{(n)} &= \cdots = y_n(t, y, \dot{y}, \ddot{y}, \ldots, \overset{n\cdot}{y}).
\end{aligned} \tag{23}$$

The differential equation for $y(t)$ is then obtained by substituting Eqs. (21), (22), and (23) into Eq. (1):

$$\hat{F}(t, y, \dot{y}, \ddot{y}, \ldots, \overset{n\cdot}{y}) = 0, \tag{24}$$

where

$$\hat{F}(t, y, \dot{y}, \ddot{y}, \ldots, \overset{n\cdot}{y}) \equiv F(x(t, y), y, y_1(t, y, \dot{y}), \ldots, y_n\big(t, y, \dot{y}, \ddot{y}, \ldots, \overset{n\cdot}{y}\big)). \tag{25}$$

Note that, from Eq. (21), the function $x(t, y)$ is homogeneous (with respect to y) of degree c, while the functions $y_k(t, y, \dot{y}, \ddot{y}, \ldots, \overset{k\cdot}{y})$ are homogeneous of degree $1 - kc$:

$$y_k(t, \alpha y, \alpha\dot{y}, \alpha\ddot{y}, \ldots, \alpha\overset{k\cdot}{y}) = \alpha^{1-kc} y_k(t, y, \dot{y}, \ddot{y}, \ldots, \overset{k\cdot}{y}), \tag{26}$$

as required for the Majorana property (14) to be satisfied. In particular from this we also deduce that the differential equation in (24) is equidimensional-in-y, since

$$\begin{aligned}
\hat{F}(t, \alpha y, \alpha\dot{y}, \alpha\ddot{y}, \ldots, \alpha\overset{n\cdot}{y}) = F(x(t, \alpha y), \alpha y, y_1(t, \alpha y, \alpha\dot{y}), \ldots, \\
\times\, y_n(t, \alpha y, \alpha\dot{y}, \alpha\ddot{y}, \ldots, \alpha\overset{n\cdot}{y}))
\end{aligned}$$

$$
\begin{aligned}
&= F(\alpha^c x(t,y), \alpha y, \alpha^{1-c} y_1(t,y,\dot{y}), \ldots, \alpha^{1-nc} \\
&\quad \times y_n(t,y,\dot{y},\ddot{y},\ldots,\overset{n\cdot}{y})) = F(x(t,y), y, \\
&\quad \times y_1(t,y,\dot{y}), \ldots, y_n(t,y,\dot{y},\ddot{y},\ldots,\overset{n\cdot}{y})) \\
&= \hat{F}(t,y,\dot{y},\ddot{y},\ldots,\overset{n\cdot}{y}),
\end{aligned} \tag{27}
$$

that is the function $\hat{F}$ satisfies the relation (11). We can then use the transformation in (12) to reduce the order of the equation. More precisely we set

$$
y(t) = e^{\int u(t)\,dt}, \tag{28}
$$

so that the t-derivatives of $y(t)$ are as follows:

$$
\begin{aligned}
\dot{y} &= uy \equiv u_1(u)y \\
\ddot{y} &= (\dot{u}+u^2)y \equiv u_2(u,\dot{u})y \\
&\vdots \\
\overset{n\cdot}{y} &= \cdots \equiv u_n\big(u,\dot{u},\ddot{u},\ldots,\overset{(n-1)\cdot}{u}\big)y.
\end{aligned} \tag{29}
$$

The unknown function is now $u(t)$ and the differential equation of order $n-1$, obeyed by this quantity, is obtained by substituting Eqs. (29) into Eq. (24):

$$
\hat{F}(t,y,u_1y,u_2y,\ldots,u_ny) = 0, \tag{30}
$$

or, by using the homogeneity of the function $\hat{F}(\hat{F}(t,y,u_1y,u_2y,\ldots,u_ny) = \hat{F}(t,1,u_1,u_2,\ldots,u_n))$ in all points where $y(t)$ is different from zero we have

$$
\hat{F}(t,1,u_1,u_2,\ldots,u_n) = 0. \tag{31}
$$

In terms of the initial function F in Eq. (1), by noting that

$$
\begin{aligned}
x(t,1) &= z(t) \\
y_1(t,1,u_1) &= \frac{u_1}{x_1(t,1,u_1)} = \frac{u}{\dot{z}+cuz} \equiv v_1(t,u) \\
y_2(t,1,u_1,u_2) &= \frac{x_1(t,1,u_1)u_2 - x_2(t,1,u_1,u_2)u_1}{x_1^3(t,1,u_1)} \\
&= \frac{\dot{z}\dot{u} - \ddot{z}u + (1-2c)\dot{z}u^2 + c(1-c)zu^3}{(\dot{z}+cuz)^3} \equiv v_2(t,u,\dot{u}) \\
&\vdots \\
y_n(t,1,u_1,u_2,\ldots,u_n) &= \cdots \equiv v_n\big(t,u,\dot{u},\ddot{u},\ldots,\overset{(n-1)\cdot}{u}\big)
\end{aligned} \tag{32}
$$

and using Eq. (25), we have the final equation for $u(t)$:

$$F\big(z(t), 1, v_1(t, u), v_2(t, u, \dot{u}), \ldots, v_n\big(t, u, \dot{u}, \ddot{u}, \ldots, \overset{(n-1)\cdot}{u}\big)\big) = 0. \tag{33}$$

Summarizing, *the parametric solution of a Majorana scale-invariant differential equation of order n has the form as in Eq. (21) with z(t) an arbitrary but given function of the parameter t and y(t) is written as in Eq. (28), where u(t) satisfies the differential equation (33) of order* $n-1$ *and the functions* $v_k(t, u, \dot{u}, \ddot{u}, \ldots, \overset{(k-1)\cdot}{u})$ *are evaluated as in Eqs. (32).*

4. THE METHOD OF THE AUXILIARY FUNCTION

Depending also on the choice for the function $z(t)$, it could happen that Eq. (33), although is of order $n-1$, is too much hard to be solved. In some cases the following procedure can be used to reduce the degree of nonlinearity of Eq. (33).

Let us perform a change of the dependent variable:

$$u \to v = \frac{u}{\dot{z} + cuz}. \tag{34}$$

In this case we have

$$\begin{aligned} v_1 &= v \\ v_2 &= (1 - cvz)\frac{\dot{v}}{\dot{z}} + (1 - c)v^2, \\ &\vdots \end{aligned} \tag{35}$$

and the function $y(t)$ in Eq. (21) is now given by

$$y(t) = e^{\int \frac{v\dot{z}}{1-cvz}\,dt}, \tag{36}$$

where $v(t)$ satisfies the differential equation of order $n-1$:

$$F\left(z, 1, v, (1 - cvz)\frac{\dot{v}}{\dot{z}} + (1 - c)v^2, \ldots\right) = 0. \tag{37}$$

5. APPLICATIONS

As an application of the Majorana method described above, let us consider the Emden–Fowler equation:

$$y'' = x^a y^b \tag{38}$$

(with a, b two real numbers), which is of particular interest in mathematical physics (Bellman, 1953). It satisfies the relation (14) with

$$c = \frac{1-b}{a+2}, \tag{39}$$

so that the method may apply only for $a \neq -2$. In this case, Eq. (33) for the Emden–Fowler equation (38) is

$$\frac{\dot{z}\dot{u} - \ddot{z}u + (1-2c)\dot{z}u^2 + c(1-c)zu^3}{(\dot{z}+cuz)^3} = z^a, \tag{40}$$

with c given in Eq. (39). After some algebra we arrive at the following first-order equation for $u(t)$:

$$\frac{du}{dt} = \alpha(t) + \beta(t)u + \gamma(t)u^2 + \delta(t)u^3, \tag{41}$$

where

$$\begin{aligned}
\alpha(t) &= z^a \dot{z}^2 \\
\beta(t) &= 3cz^{a+1}\,\dot{z} + \frac{\ddot{z}}{z} \\
\gamma(t) &= 3c^2\,z^{a+2} + 2c - 1 \\
\delta(t) &= c^3\frac{z^{a+3}}{\dot{z}} + c(c-1)\frac{z}{\dot{z}}.
\end{aligned} \tag{42}$$

With the Majorana method we have thus transformed the Emden–Fowler equation (38) into an Abel equation (41) of the first kind. Depending on the particular problem to be solved, the final equation can be further simplified with an appropriate choice for $z(t)$, which, however, cannot be chosen equal to a constant (in this case $\dot{z} = 0$ and Eq. (40) would not be a differential equation for u).

A relevant case is that of Emden–Fowler equation with $b = 1$ for which, from Eq. (39), we have $c = 0$ and Eq. (41) reduces to a simpler Riccati equation:

$$\frac{du}{dt} = z^a\,\dot{z}^2 + \frac{\ddot{z}}{z}\,u - u^2, \tag{43}$$

which for $z(t) = t$ becomes

$$\frac{du}{dt} = t^a - u^2. \tag{44}$$

Another interesting particular case is that of Thomas–Fermi equation, which is an Emden–Fowler equation with $a = -1/2$, $b = 3/2$:

$$y'' = \frac{y^{3/2}}{\sqrt{x}}. \tag{45}$$

The corresponding first-order equation (41) for $u(t)$, choosing

$$z(t) = [12(1-t)]^{2/3}, \tag{46}$$

is

$$\frac{du}{dt} = \frac{16}{3(1-t)} + \left(8 + \frac{1}{3(1-t)}\right) u + \left(\frac{7}{3} - 4t\right) u^2 - \frac{2}{3} t(1-t)\, u^3. \tag{47}$$

This equation was obtained by Majorana (Esposito *et al.*, in press) in studying the Thomas–Fermi equation. Emden–Fowler equations can also be analyzed by using the method of the auxiliary function outlined in the previous section. In this case, Eq. (37) for $v(t)$ becomes

$$\frac{dv}{dt} = \frac{\dot{z}[z^a - (1-c)v^2]}{1 - cvz}, \tag{48}$$

where c is given in Eq. (39). Note that in this case also we cannot choose $z(t) =$ constant ($\dot{z} = 0$) from Eq. (36).

Although Eqs. (48) and (41) are different, in the particular case with $b = 1$, and thus $c = 0$, we again obtain a Riccati equation:

$$\frac{dv}{dt} = \dot{z}(z^a - v^2). \tag{49}$$

Instead, following the method of the auxiliary function with

$$z(t) = 12^{2/3}\, t^2, \tag{50}$$

the Thomas–Fermi equation can be transformed into the first-order equation

$$\frac{d\tilde{v}}{dt} = 8 \frac{t\tilde{v}^2 - 1}{1 - t^2\tilde{v}}, \tag{51}$$

where, for simplicity, we have set $\tilde{v} = -4(12^{-1/3}\, v)$. Equation (51) has been solved using series expansion by Majorana (Esposito *et al.*, in press), and this leads to a semianalytic general solution for the Thomas–Fermi equation (for details, see Esposito, in press).

6. CONCLUSIONS AND OUTLOOK

In this paper we have generalized a result, derived by Majorana (Esposito, in press; Esposito *et al.*, in press) for solving the Thomas–Fermi equation, to a wide class of (ordinary) differential equations, that of Majorana scale-invariant equations, as defined in Section 3. We have shown that the search for the parametric solution of such equations of order n can be restricted to that for the solution of a differential equation of order $n-1$ (and to the computation of one integral involving this solution). However, the main result of this paper is not this proposition, which is a direct consequence of known results, but rather the method and the transformations used to obtain the lower order equation. This has been outlined in Section 3 and the equation considered has been formally written in Eq. (33). In some case this differential equation could be highly nonlinear, and so in Section 4 we have developed a variant of the method mentioned above, employing a further transformation for the dependent variable involved. Some other simplifications, depending on the particular problem considered, can be achieved with a suitable choice for the arbitrary function $z(t)$ present in the parametric solution for the differential equation.

As an illustration, both methods have been applied to reduce the order of Emden–Fowler equations in Section 5 and, as a particular case, Thomas–Fermi equation has been considered as well. Remarkably, by using the method of Section 3, we have shown that all second-order Emden–Fowler equations can be transformed into first-order Abel equations of the first kind. Instead, by using the method of the auxiliary function reported in Section 4, the Thomas–Fermi equation can be transformed into a suitable first-order equation which can be solved by series expansion.

We believe that the transformation methods presented here deserve further attention in view of their potential applications to scale-invariant differential equations which are of interest for mathematical physics.

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